MALAYSIAN JOURNAL OF SCIENCE
MJS
ISSN 1394-3065

Regular Issue

Malaysian Journal of Science 44 (1): 70-86 (March 2025) https://mjs.um.edu.my

On the Kronecker Structure of linearization of Cubic Two-Parameter Eigenvalue Problems

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Abstract: Linearization is a conventional approach to studying matrix polynomials of the form $P(\lambda) := \sum_{j=1}^{k} \lambda^j A_j$, where $A_j \in \mathbb{C}^{n\times n}$. It converts the matrix polynomial $P(\lambda)$ into a matrix pencil of the form $L(\lambda) := A + \lambda B$ of high dimension, where A and B are matrices over $\mathbb{C}$, and λ is the spectral parameter. In this paper, we consider Cubic two-parameter eigenvalue problems ($\mathbb{CTEP}$) and study their three different linearization processes. Using linearization techniques, a $\mathbb{CTEP}$ is first converted into a linear two-parameter eigenvalue problem ($\mathbb{L}2\mathbb{EP}$) with coefficient matrices of different sizes. The main advantage of these linearizations lies in the fact that, after transforming them into suitable linearized forms, existing numerical techniques for linear multiparameter eigenvalue problems ($\mathbb{LMEP}$) can be applied to solve the $\mathbb{CTEP}$ without solving the original problem. While solving $\mathbb{CTEP}$ by formulating suitable linearizations, several transformations are generally used. This study reports on these transformations, which have not been studied completely due to the complexity of their Kronecker structures. The ranks of the associated Delta matrices are also calculated in a detailed manner to bring out the benefits of using the Tracy-Singh product over others.

Keywords: *Cubic two-parameter eigenvalue problem, Linear two-parameter eigenvalue problem, linearization, matrix polynomial, Tracy-Singh product.*

1. Introduction

One-parameter matrix polynomials arise in many physical applications and have received significant attention from researchers (Dmytryshyn et al., 2020; Fabbender & Saltenberger, 2018; Gohberg et al., 2009). However, the literature on two-parameter matrix polynomials remains limited (Hochstenbach et al., 2015; Jarlebring et al., 2009). The standard form of a two-parameter matrix polynomial of degree k is given by

$$\mathbb{P}(\lambda,\mu) := \sum_{i=0}^{k}\sum_{j=0}^{k-i} \lambda^i \mu^j P_{ij} = \sum_{j=0}^{k} \mathbb{E}_j(\lambda,\mu), \qquad (1)$$

where $P_{ij} \in \mathbb{C}^{n\times n}$, $\lambda,\mu \in \mathbb{C}$ are spectral parameters and $\mathbb{E}_j(\lambda,\mu)$ is a homogeneous matrix polynomial of degree j such that,

$$\mathbb{E}_j(\lambda,\mu) := \sum_{l=0}^{j} \lambda^{j-l} \mu^l P_{jl} \qquad (2)$$

The standard form of the Polynomial two-parameter eigenvalue problem ($\mathbb{PTEP}$), which is the generalization of the Polynomial eigenvalue problem ($\mathbb{PEP}$), comprises two bivariate matrix polynomials of the form

$$\begin{aligned}\mathbb{P}_1(\lambda,\mu)x_1 &:= \sum_{i=0}^{k}\sum_{j=0}^{k-i} \lambda^i \mu^j A_{ij} x_1 = 0,\\ \mathbb{P}_2(\lambda,\mu)x_2 &:= \sum_{i=0}^{k}\sum_{j=0}^{k-i} \lambda^i \mu^j B_{ij} x_2 = 0,\end{aligned} \qquad (3)$$

where $A_{ij} \in \mathbb{C}^{n_1\times n_1}$; $B_{ij} \in \mathbb{C}^{n_2\times n_2}$ and $x_i \in \mathbb{C}^{n_i}$, $i := 1,2$ are non zero vectors. The problem is to find the scalars $\lambda,\mu \in \mathbb{C}$ and the corresponding non zero vectors $x_i \in \mathbb{C}^{n_i}$, $i := 1:2$ such that $\mathbb{P}_j(\lambda,\mu)x_j := 0$. The pair $(\lambda,\mu) \in \mathbb{C}^2$ is called the eigenvalue and the corresponding tensor product $x := x_1 \otimes x_2$ is called the right eigenvectors. Similarly, a tensor product $v_1 \otimes v_2$ is called a left eigenvector of the $\mathbb{PTEP}$ if $v_i \neq 0$; $i := 1:2$, satisfies $v_i^* \mathbb{P}_i(\lambda,\mu) = 0$. For $k = 2$, the Equation defined in (3) is reduced to a Quadratic two-parameter eigenvalue problem ($\mathbb{QTEP}$), and for $k = 3$, it is reduced to a $\mathbb{CTEP}$.

$\mathbb{PTEP}$ topic emerges in the analysis of critical delay differential equations (Jarlebring & Hochstenbach, (2009); Meerbergen et al., (2013)). For instance, the neutral commensurate differential equations (Hochstenbach et al., (2005)) with multiple delays (m>1) (Hochstenbach et al., (2015)). Two methodological approaches exist to address this phenomenon. The first approach enables parametrization of surfaces or curves corresponding to the critical delay using m-1 independent variables. The second approach posits that the delays are commensurate, functioning as integer multiples of a specific delay value τ≥0. The delay

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Received: August 18, 2023
Accepted: January 14, 2024
Published: March 31, 2025

DOI:https//doi.org/10.22452/mjs.vol44no1.9
Malaysian Journal of Science 44(1): 70-86 (March 2025)

differential equation featuring commensurate delays, wherein delays operate as integer multiples of a delay value τ, is

$$N_0\dot{x}(t) = \sum_{k=0}^{m} M_k\, x(t-\tau k), \qquad (4)$$

where $M_k, N_0 \in \mathbb{C}^{n\times n}$. The associated eigenvalue problem is,

$$\left(\sum_{k=0}^{m} e^{k\tau\lambda} M_k - \lambda N_0\right)x = 0. \qquad (5)$$

In stability analysis, the purely imaginary eigenvalues are preferred. For this reason, consider $\lambda = i\omega$ and $\mu = e^{i\tau\omega}$. Their conjugates are $\bar{\lambda} = -\lambda$ and $\bar{\mu} = \mu^{-1}$. Taking complex conjugates of the Equation (5) and rearranging terms we obtain,

$$\begin{aligned} -\bar{M}_m x &= \lambda\mu^m \bar{N}_0 x + \sum_{k=1}^{m} \mu^k\, \bar{M}_{m-k} x, \\ M_0 y &= \lambda N_0 y - \sum_{k=1}^{m} \mu^k\, \bar{M}_k y. \end{aligned} \qquad (6)$$

Equation (6) motivates the study of the eigenvalue problem in the following form

$$\begin{aligned} M_1 x &= \lambda\sum_{k=0}^{m} \mu^k\, N_{1,k} x + \sum_{k=1}^{m} \mu^k\, C_{1,k} x, \\ M_2 y &= \lambda\sum_{k=0}^{m} \mu^k\, N_{2,k} y + \sum_{k=1}^{m} \mu^k\, C_{2,k} y, \end{aligned} \qquad (7)$$

which is the general form of $\mathbb{PTEP}$. $\mathbb{PTEP}$ also arises in the study of bivariate polynomials (Plestenjak, 2017; Plestenjak & Hochstenbach, 2016), and the references therein.

There are two types of numerical approaches for solving $\mathbb{PTEP}$: those that deal directly with the problem and those that compute eigenvalues of linearized forms. The usual method to solve the $\mathbb{PTEP}$ defined in (3) is by linearizing it into an $\mathbb{L}2\mathbb{EP}$ of larger dimension. The linearized version of problem (3) is singular and can be solved by adopting the method proposed in (Dooren, 1997; Muhič & Plestenjak, 2009; Košir & Plestenjak, 2022). Moreover, the Jacobi-Davidson method developed in Hochstenbach et al. (2015) can be applied directly to $\mathbb{PTEP}$ instead of the linearized problem. Linearization is a classical approach to investigating the $\mathbb{PEP}$. Details on linearization of one-parameter matrix polynomials are found in the works of Mackey et al. (2006), Bueno et al. (2018), Das and Alam (2019), Das (2020), Higham et al. (2006), and Lancaster (2008), and the references therein. Literature on linearizations for quadratic matrix polynomials is found in Kressner and Glibić (2023) and Lancaster and Zaballa (2021). The linearization process influences the sensitivity of eigenvalues. Therefore, it is important to identify potential linearizations and study their constructions. The linearized form of two-parameter polynomials has a somewhat complicated structure compared to the one-parameter case. Literature on linearization of $\mathbb{QTEP}$ is found in the works of Tisseur and Meerbergen (2001), Muhič and Plestenjak (2010), and Hochstenbach et al. (2012), and numerical methods are found in Plestenjak (2016) and Dong (2022). In this paper, we provide a general framework for the canonical structure of the linearized form of $\mathbb{CTEP}$, which can be considered a continuing thread to study the general $\mathbb{PTEP}$ of degree k.

This paper is organized as follows: Section 2 contains basic preliminaries. Section 3 contains the problem formulation and its basic theory. Section 4 contains a unified framework on different linearization techniques of $\mathbb{CTEP}$. In Section 5, the ranks of delta matrices involved in $\mathbb{CTEP}$ are derived. A numerical example is presented in Section 6 to compare the linearization classes, and finally, in Section 7, a conclusion is drawn on the whole work.

2. Preliminaries

The following basic definitions and results are applied throughout the paper: $A \in \mathbb{C}^{n_1\times n_2}$ is the matrix of size $n_1 \times n_2$ over $\mathbb{C}$. A^{-1}, A^T and A^* represents the inverse, transpose and conjugate transpose of the matrix A, respectively. The Euclidean norm of the matrix A is denoted by $\|A\|$ and the standard Kronecker product is denoted by $\otimes$.

Definition 1. (Henderson et al., (1983)) *The Kronecker Product* ($\otimes$) *for two matrices A and B is defined as* $A \otimes B = \{a_{ij}B\}$, *where* a_{ij} *are the elements in* i^{th} *row and* j^{th} *column of the matrix A.*

Definition 2. (Tracy & Singh, (1972)) *Tracy–Singh product of partitioned matrices: Let an* $m \times n$ *matrix A be partitioned into the* $m_i \times n_j$ *blocks* A_{ij} *and a* $p \times q$ *matrix B into the* $p_k \times q_l$ *blocks* B_{kl} *such that* $m = \sum_{i=1}^{r} m_i$, $n = \sum_{j=1}^{s} n_j$, $p = \sum_{k=1}^{t} p_k$, $q = \sum_{l=1}^{u} q_l$. *The Tracy–Singh product* $A \odot B$ *is a* $mp \times nq$ *matrix, defined as* $A \odot B = \left(A_{ij} \odot B\right)_{ij} = \left(\left(A_{ij} \otimes B_{kl}\right)_{kl}\right)_{ij}$, *where the (ij)th block of the product is the* $m_i p \times n_j q$ *matrix* $A_{ij} \odot B$, *of which the (kl)th subblock equals the* $m_i p_k \times n_j q_l$ *matrix* $A_{ij} \otimes B_{kl}$.

For example, if we take $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ *and* $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$; *then the Tracy-Singh product is defined as,*

$$A \odot B = \begin{pmatrix} A_{11} \odot B & A_{12} \odot B \\ A_{21} \odot B & A_{22} \odot B \end{pmatrix} = \begin{pmatrix} A_{11} \otimes B_{11} & A_{11} \otimes B_{12} & A_{12} \otimes B_{11} & A_{12} \otimes B_{12} \\ A_{11} \otimes B_{21} & A_{11} \otimes B_{22} & A_{12} \otimes B_{21} & A_{12} \otimes B_{22} \\ A_{21} \otimes B_{11} & A_{21} \otimes B_{12} & A_{22} \otimes B_{11} & A_{22} \otimes B_{12} \\ A_{21} \otimes B_{21} & A_{21} \otimes B_{22} & A_{22} \otimes B_{21} & A_{22} \otimes B_{22} \end{pmatrix}$$

DOI:https//doi.org/10.22452/mjs.vol44no1.9
Malaysian Journal of Science 44(1): 70-86 (March 2025)

We will denote the Tracy-Singh Product by a map TSP.

Definition 3. (Muhič & Plestenjak, (2010))) *Tracy-Singh reordering of two block matrices* A *and* B *is given by a map TSR, such that $TSR(A \otimes B) = A \odot B$, i.e., we reorder the columns and rows of the Kronecker product $A \otimes B$, to obtain the Tracy-Singh product.*

Theorem 1. (Tracy & Jinadasa, (1989)) *When A and B can be partitioned into equal-sized blocks, then Tracy-Singh product $A \odot B$ and the Kronecker product $A \otimes B$ are permutation equivalent.*

Definition 4. (Hochstenbach, (2003)) *The generalized eigenvalue problem ($\mathbb{GEP}$) is to find the pair (λ, x) that satisfies the matrix equation of the form $Ax = \lambda Bx$, where A and B are any matrices over $\mathbb{C}$, x is a non-zero vector, and λ is the spectral parameter.*

Definition 5. (Atkinson, (1972)) $\mathbb{LMEP}$ *is to find the scalars $\lambda_i \in \mathbb{C}$ and the corresponding non-zero vectors $x_i \in \mathbb{C}^{m_i}$ such that,*

$$\mathbb{W}(X, \Omega) = \begin{cases} \mathbb{W}_1(\lambda)x_1 \\ \dots \\ \mathbb{W}_n(\lambda)x_n \\ \frac{1}{2}(x_1^* x_1 - 1) \\ \dots \\ \frac{1}{2}(x_n^* x_n - 1) \end{cases} = 0, \qquad (8)$$

where $\mathbb{W}_i(\lambda) := -A_{i0} + \sum_{j=1}^{n} \lambda_j A_{ij}$; $A_{ij} \in \mathbb{C}^{m_i \times m_i}$; $i := 1{:}n$; $j := 0{:}n$. $\mathbb{L2EP}$ *being the special case of* $\mathbb{LMEP}$ when $\mathrm{n} = 2$.

Definition 6. (Hochstenbach et al., (2012)) *Let $\mathbb{Q}(\lambda, \mu) := \sum_{i=0}^{k} \sum_{j=0}^{k-i} \lambda^i \mu^j P_{ij}$ be any $n \times n$ matrix polynomial. Then, an $ln \times ln$ linear matrix polynomial $\mathbb{L}(\lambda, \mu) = L_0 + \lambda L_1 + \mu L_2$ is a linearization of $\mathbb{Q}(\lambda, \mu)$ if there exist polynomials $\mathcal{M}(\lambda, \mu)$ and $\mathcal{N}(\lambda, \mu)$, whose determinant is a non-zero constant independent of λ and μ, such that* $\begin{bmatrix} \mathbb{Q}(\lambda, \mu) & 0 \\ 0 & I_{(l-1)n} \end{bmatrix} = \mathcal{M}(\lambda, \mu)\mathbb{L}(\lambda, \mu)\mathcal{N}(\lambda, \mu)$.

3. General Theory of $\mathbb{CTEP}$

The standard form of $\mathbb{CTEP}$, which is being the special case $\mathbb{PTEP}$ when $k = 3$, is given by

$$\begin{aligned} \mathbb{P}_1(\lambda, \mu)x_1 = 0, \\ \mathbb{P}_2(\lambda, \mu)x_2 = 0, \end{aligned} \qquad (9)$$

where

$\mathbb{P}_1(\lambda, \mu) = \lambda^3 A_{30} + \lambda^2 \mu A_{21} + \lambda \mu^2 A_{12} + \mu^3 A_{03} + \lambda^2 A_{20} + \lambda \mu A_{11} + \mu^2 A_{02} + \lambda A_{10} + \mu A_{01} + A_{00}$;

$\mathbb{P}_2(\lambda, \mu) = \lambda^3 B_{30} + \lambda^2 \mu B_{21} + \lambda \mu^2 B_{12} + \mu^3 B_{03} + \lambda^2 B_{20} + \lambda \mu B_{11} + \mu^2 B_{02} + \lambda B_{10} + \mu B_{01} + B_{00}$; and A_{ij}, B_{ij} are $n \times n$ matrices over $\mathbb{C}$; $i := 1{:}2$, $j := 0{:}9$ such that at least one of the matrices $A_{30}, A_{03}, B_{30}, B_{03}, A_{21}, A_{12}, B_{21}, B_{12}$ is nonzero.

The $\mathbb{CTEP}$ appears in prior work (Muhič & Plestenjak, 2010) (Example 20), where the problem is linearized into a $\mathbb{L2EP}$. However, the authors did not provide proof of the Kronecker structure involved in theory, similar to the quadratic case, due to the complexity arising in the respective Kronecker canonical structure. For a given $\mathbb{CTEP}$ defined in (9), we investigate the $\mathbb{L2EP}$,

$$\begin{aligned} \mathbb{L}^{(1)}(\lambda, \mu)w_1 = \left(\mathbb{L}_0^{(1)} + \lambda \mathbb{L}_1^{(1)} + \mu \mathbb{L}_2^{(1)}\right) w_1 = 0 \\ \mathbb{L}^{(2)}(\lambda, \mu)w_2 = \left(\mathbb{L}_0^{(2)} + \lambda \mathbb{L}_1^{(2)} + \mu \mathbb{L}_2^{(2)}\right) w_2 = 0 \end{aligned} \qquad (10)$$

where $w_i \in \mathbb{C}^{6n}$ 0; $\mathbb{L}_j^{(i)} \in \mathbb{C}^{6n \times 6n}$, $i := 1{:}2, j = 0{:}2$, such that (10) agrees with the eigenvalues of (9). Converting the problem into a system of joint $\mathbb{GEP}$ in the tensor product space is considered the de facto method, known as the Delta method (Atkinson, 1972) for spectral analysis of the problem. The equivalence between the problem $\mathbb{L2EP}$ and the corresponding joint $\mathbb{GEP}$ can be established by transforming the problem into a commuting pair of specific operator matrices with the following operator determinants,

$$\Delta_0 := \mathbb{L}_1^{(1)} \otimes \mathbb{L}_2^{(2)} - \mathbb{L}_2^{(1)} \otimes \mathbb{L}_1^{(2)}$$

$$\Delta_1 := \mathbb{L}_2^{(1)} \otimes \mathbb{L}_0^{(2)} - \mathbb{L}_0^{(1)} \otimes \mathbb{L}_2^{(2)}; \Delta_2 := \mathbb{L}_0^{(1)} \otimes \mathbb{L}_1^{(2)} - \mathbb{L}_1^{(1)} \otimes \mathbb{L}_0^{(2)} \qquad (11)$$

Then each Δ_i, $i := 1{:}2$ is $N \times N$ matrices, where $N := 36n^2$. The system (10) is referred to as singular or nonsingular, according to the operator matrix Δ_0 specified in Equation (11). The proof for the singularity of Δ_0 has been discussed in Section 5, along with the other two operator matrices Δ_1 and Δ_2 using Tracy-Singh product. For spectral analysis, the linear $\mathbb{PTEP}$ is generally considered as nonsingular and a commuting tuple of the form $\Gamma := (\Gamma_1, \Gamma_2)$ is used, where $\Gamma_i := \Delta_0^{-1}\Delta_i$; $i := 1{,}2$ and is equivalent to a system of joint $\mathbb{GEP}$ of the form given by,

$$\Delta_j u = \lambda_j \Delta_0 u; \quad j := 1, 2; \qquad (12)$$

where $u = w_1 \otimes w_2 \in \mathbb{C}^N$ is a decomposable tensor. System (10) is called linearization of $\mathbb{CTEP}$ defined in (9).

Theorem 2. (Atkinson, (1972); Muhič & Plestenjak, (2010)) *For given values α_0, α_1 and α_2, the homogeneous problem,*

$$\left(\eta_0 \mathbb{L}_0^{(1)} + \eta_1 \mathbb{L}_1^{(1)} + \eta_2 \mathbb{L}_2^{(1)}\right) w_1 = 0$$
$$\left(\eta_0 \mathbb{L}_0^{(2)} + \eta_1 \mathbb{L}_1^{(2)} + \eta_2 \mathbb{L}_2^{(2)}\right) w_2 = 0 \qquad (13)$$

satisfies the following equivalent conditions

1. *The matrix $\Delta = \sum_{i=0}^{2} \alpha_i \Delta_i$ is singular.*

2. *There exists an eigenvalue (η_0, η_1, η_2) of the system (12) such that $\sum_{i=0}^{2} \eta_i \alpha_i = 0$*

The same result for nonsingularity has been stated in the following way also.

Theorem 3. (Atkinson, (1972); Muhič & Plestenjak, (2010)) *The homogeneous* $\mathbb{LMEP}$

$$\sum_{j=0}^{k} \eta_j A_{ij} x_i = 0 \qquad (14)$$

where $A_{ij} \in \mathbb{C}^{n_i \times n_i}$, for $i = 1, \dots, k$ and $j = 0, \dots, k$ is said to be nonsingular if there exists a nonsingular linear combination of the operator determinants Δ_i's, i.e., $\Delta = \sum_{i=0}^{k} \alpha_i \Delta_i$. This is equivalent to the condition that, if $\eta = (\eta_0, \eta_1, \dots, \eta_k)$ is an eigenvalue of (14), then $\sum_{i=0}^{k} \alpha_i \eta_i \neq$ 0.

Theorem 4. (Cox et al., (2005)) *(Bezout's theorem) Two projective curves of orders n and m with no common component has precisely nm points of intersection counting multiplicities.*

4. Linearization of $\mathbb{CTEP}$

In this section, we present three different types of linearization techniques of $\mathbb{CTEP}$. Two of them are the general linearization, resulting in a singular $\mathbb{L2EP}$, with coefficient matrices of size $6n \times 6n$ and $9n \times 9n$, respectively. The singularity conditions for the associated $\mathbb{GEP}$ is shown with the help of Tracy-Singh reordering in Δ_i's. The third type of linearization is done by replacing nonlinear terms with new variables, which formulates a nonsingular linear nine-parameter eigenvalue problem ($\mathbb{L9EP}$) so that more efficient methods for solving nonsingular problems can be applied in this case.

Standard Linearization

For a given $\mathbb{CTEP}$, and by following definition 5, we can linearize the $\mathbb{CTEP}$ into a $\mathbb{L2EP}$ (Muhič & Plestenjak, 2010) of the form,

$$\left(\begin{bmatrix} A_{00} & A_{10} & A_{01} & A_{20} & A_{11} & A_{02} \\ 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & 0 & A_{30} & A_{21} & A_{12} \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & A_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ \lambda x_1 \\ \mu x_1 \\ \lambda^2 x_1 \\ \lambda\mu x_1 \\ \mu^2 x_1 \end{bmatrix} = 0,$$

$$\left(\begin{bmatrix} B_{00} & B_{10} & B_{01} & B_{20} & B_{11} & B_{02} \\ 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & 0 & B_{30} & B_{21} & B_{12} \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & B_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} x_2 \\ \lambda x_2 \\ \mu x_2 \\ \lambda^2 x_2 \\ \lambda\mu x_2 \\ \mu^2 x_2 \end{bmatrix} = 0. \qquad (15)$$

Comparing Equation (15) with that of the Equation (10), we have,

$$\mathbb{L}_0^{(1)} = \begin{bmatrix} A_{00} & A_{10} & A_{01} & A_{20} & A_{11} & A_{02} \\ 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix}, \quad \mathbb{L}_1^{(1)} = \begin{bmatrix} 0 & 0 & 0 & A_{30} & A_{21} & A_{12} \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbb{L}_2^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & A_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}, \mathbb{L}_0^{(2)} = \begin{bmatrix} B_{00} & B_{10} & B_{01} & B_{20} & B_{11} & B_{02} \\ 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix},$$

DOI:https//doi.org/10.22452/mjs.vol44no1.9
Malaysian Journal of Science 44(1): 70-86 (March 2025)

$$\mathbb{L}_1^{(2)}=\begin{bmatrix}0&0&0&B_{30}&B_{21}&B_{12}\\ I&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&I&0&0&0&0\\ 0&0&I&0&0&0\\ 0&0&0&0&0&0\end{bmatrix},\ \mathbb{L}_2^{(2)}=\begin{bmatrix}0&0&0&0&0&B_{03}\\ 0&0&0&0&0&0\\ I&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&I&0&0&0\end{bmatrix}.$$

For the first Equation in (15), consider,

$$\Lambda=\begin{bmatrix}1\\ \lambda\\ \mu\\ \lambda^2\\ \lambda\mu\\ \mu^2\end{bmatrix},\ \text{and}\ w_1=\Lambda\otimes x_1=\begin{bmatrix}x_1\\ \lambda x_1\\ \mu x_1\\ \lambda^2x_1\\ \lambda\mu x_1\\ \mu^2x_1\end{bmatrix}.$$

Thus, x_1 is an eigenvector corresponding to the eigenvalue (λ,μ) of $\mathbb{P}_1(\lambda,\mu)$ from Equation (9) if and only if $w_1=\Lambda\otimes x_1$ is an eigenvector corresponding to the eigenvalue (λ,μ) of $\mathbb{L}^{(1)}(\lambda,\mu)$ from Equation (10). Now, using the definition 6, we demonstrate that $\mathbb{L}^{(1)}(\lambda,\mu)$ is a linearization of $\mathbb{P}_1(\lambda,\mu)$. For that define

$$\mathcal{N}(\lambda,\mu)=\begin{bmatrix}I_n&0&0&0&0&0\\ \lambda I_n&0&0&0&0&I_n\\ \mu I_n&0&0&0&I_n&0\\ \lambda^2I_n&0&0&I_n&0&0\\ \lambda\mu I_n&0&I_n&0&0&0\\ \mu^2I_n&I_n&0&0&0&0\end{bmatrix},$$

$$\mathcal{M}(\lambda,\mu)=\begin{bmatrix}I_n&S_1(\lambda,\mu)&S_2(\lambda,\mu)&A_{20}+\lambda A_{30}&A_{11}+\lambda A_{21}&A_{02}+\lambda A_{12}+\mu A_{03}\\ 0&0&\mu I_n&0&0&-I_n\\ 0&0&\lambda I_n&0&-I_n&0\\ 0&\lambda I_n&0&-I_n&0&0\\ 0&0&-I_n&0&0&0\\ 0&-I_n&0&0&0&0\end{bmatrix},$$

where $S_1(\lambda,\mu)=A_{10}+\lambda A_{20}+\lambda^2A_{30}$ and $S_2(\lambda,\mu)=A_{01}+\lambda A_{11}+\lambda^2A_{21}+\mu A_{02}+\lambda\mu A_{12}+\mu^2A_{03}$. Then we can check that,

$$\mathcal{M}(\lambda,\mu)\mathbb{L}^{(1)}(\lambda,\mu)\mathcal{N}(\lambda,\mu)=\begin{pmatrix}\mathbb{P}_1(\lambda,\mu)&0\\ 0&I_{5n}\end{pmatrix}.$$

Thus, we have $\det\mathbb{P}_1(\lambda,\mu)=\alpha\det\mathbb{L}^{(1)}(\lambda,\mu)$, for some $\alpha\neq0$. This indicates that $\mathbb{L}^{(1)}(\lambda,\mu)$ is a llinearization of $\mathbb{P}_1(\lambda,\mu)$ and it preserves the eigenvalues of $\mathbb{P}_1(\lambda,\mu)$. Similarly, we can also check for the second Equation of (15).

Khazanov Linearization

This approach was presented by Khazanov (2007). In this approach, we first write $\mathbb{P}_1(\lambda,\mu)$ as a polynomial in λ.

$$(\lambda^3A_{30}+\lambda^2(\mu A_{21}+A_{20})+\lambda(\mu^2A_{12}+\mu A_{11}+A_{10})+(\mu^3A_{03}+\mu^2A_{02}+\mu A_{01}+A_{00})x_1=0. \qquad (16)$$

Now, by using the first companion form, Equation (16) can be linearized as,

$$\left(\lambda\begin{bmatrix}0&0&A_{30}\\ 0&I&0\\ I&0&0\end{bmatrix}+\begin{bmatrix}\mu^3A_{03}+\mu^2A_{02}+\mu A_{01}+A_{00}&\mu^2A_{12}+\mu A_{11}+A_{10}&\mu A_{21}+A_{20}\\ 0&0&-I\\ 0&-I&0\end{bmatrix}\right)\begin{bmatrix}x_1\\ \lambda x_1\\ \lambda^2x_1\end{bmatrix}=0. \qquad (17)$$

Now, by considering the polynomial in μ, we obtain

$$\left(\mu^3\begin{bmatrix}A_{03}&0&0\\ 0&0&0\\ 0&0&0\end{bmatrix}+\mu^2\begin{bmatrix}A_{02}&A_{12}&0\\ 0&0&0\\ 0&0&0\end{bmatrix}+\mu\begin{bmatrix}A_{01}&A_{11}&A_{21}\\ 0&0&0\\ 0&0&0\end{bmatrix}+\begin{bmatrix}A_{00}&A_{10}&\lambda A_{30}+A_{20}\\ 0&\lambda I&-I\\ \lambda I&-I&0\end{bmatrix}\right)\begin{bmatrix}x_1\\ \lambda x_1\\ \lambda^2x_1\end{bmatrix}=0. \qquad (18)$$

By using the first companion form for linearization, we have

DOI:https//doi.org/10.22452/mjs.vol44no1.9
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$$\left(\mu\begin{bmatrix} 0 & 0 & \begin{bmatrix} A_{03} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ 0 & I_{3n} & 0 \\ I_{3n} & 0 & 0 \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} A_{00} & A_{10} & \lambda A_{30}+A_{20} \\ 0 & \lambda I & -I \\ \lambda I & -I & 0 \end{bmatrix} & \begin{bmatrix} A_{01} & A_{11} & A_{21} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} A_{02} & A_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ 0 & 0 & -I_{3n} \\ 0 & -I_{3n} & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ \lambda x_1 \\ \lambda^2 x_1 \\ \mu x_1 \\ \lambda\mu x_1 \\ \lambda^2\mu x_1 \\ \mu^2 x_1 \\ \lambda\mu^2 x_1 \\ \lambda^2\mu^2 x_1 \end{bmatrix} = 0, \quad (19)$$

which can also be rewritten as

$$\left(\mu\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & A_{03} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} A_{00} & A_{10} & \lambda A_{30}+A_{20} & A_{01} & A_{11} & A_{21} & A_{02} & A_{12} & 0 \\ 0 & \lambda I_n & -I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda I_n & -I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_n \\ 0 & 0 & 0 & -I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_n & 0 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ \lambda x_1 \\ \lambda^2 x_1 \\ \mu x_1 \\ \lambda\mu x_1 \\ \lambda^2\mu x_1 \\ \mu^2 x_1 \\ \lambda\mu^2 x_1 \\ \lambda^2\mu^2 x_1 \end{bmatrix} = 0. \quad (20)$$

This is equivalent to

$$\lambda\begin{bmatrix} 0 & 0 & A_{30} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \mu\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & A_{03} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} A_{00} & A_{10} & A_{20} & A_{01} & A_{11} & A_{21} & A_{02} & A_{12} & 0 \\ 0 & 0 & -I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_n \\ 0 & 0 & 0 & -I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_n & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ \lambda x_1 \\ \lambda^2 x_1 \\ \mu x_1 \\ \lambda\mu x_1 \\ \lambda^2\mu x_1 \\ \mu^2 x_1 \\ \lambda\mu^2 x_1 \\ \lambda^2\mu^2 x_1 \end{bmatrix} = 0. \quad (21)$$

This linearization is called Khazanov Linearization. Proceeding similarly for $\mathbb{P}_2(\lambda, \mu)$, the respective linearization can be obtained. In the place of the first companion form of linearizations in Equations (17) and (19), if we use different forms of linearizations, we obtain further linearizations with $9n \times 9n$ matrices. The size of the matrices in the Khazanov linearization in Equation (21) is $9n \times 9n$, which is larger than that of the Standard linearization ($6n \times 6n$). Thus, the Khazanov linearization is numerically less efficient than that of the Standard linearization. Moreover, one can further deduce the standard linearization from the Khazanov linearization of (21).

DOI:https//doi.org/10.22452/mjs.vol44no1.9
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Linearization Like Method

Consider the $\mathbb{CTEP}$ defined in (9), we introduce the new variables $\alpha=\lambda^3$, $\beta=\lambda^2\mu$, $\gamma=\lambda\mu^2$, $\delta=\mu^3$, $\eta=\lambda^2$, $\nu=\lambda\mu$ and $\sigma=\mu^2$. The $\mathbb{CTEP}$ can be rewritten as a linear $\mathbb{L}9\mathbb{EP}$ as follows:

$$
\begin{gathered}
(\alpha A_{30}+\beta A_{21}+\gamma A_{12}+\delta A_{03}+\eta A_{20}+\nu A_{11}+\sigma A_{02}+\lambda A_{10}+\mu A_{01}+A_{00})x_1=0,\\
(\alpha B_{30}+\beta B_{21}+\gamma B_{12}+\delta B_{03}+\eta B_{20}+\nu B_{11}+\sigma B_{02}+\lambda B_{10}+\mu B_{01}+B_{00})x_2=0,\\
\left(\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+\lambda\begin{bmatrix}0&0&-1\\-1&0&0\\0&0&0\end{bmatrix}+\alpha\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\right)\begin{bmatrix}1\\\lambda\\\lambda^2\end{bmatrix}=0,\\
\left(\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+\lambda\begin{bmatrix}0&0&0\\-1&0&0\\0&0&0\end{bmatrix}+\mu\begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}+\beta\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\right)\begin{bmatrix}1\\\lambda\\\lambda^2\end{bmatrix}=0,\\
\left(\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+\lambda\begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}+\mu\begin{bmatrix}0&0&0\\-1&0&0\\0&0&0\end{bmatrix}+\beta\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\right)\begin{bmatrix}1\\\mu\\\mu^2\end{bmatrix}=0,\\
\left(\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+\mu\begin{bmatrix}0&0&-1\\-1&0&0\\0&0&0\end{bmatrix}+\delta\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\right)\begin{bmatrix}1\\\mu\\\mu^2\end{bmatrix}=0,\\
\left(\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+\lambda\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}+\eta\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\right)\begin{bmatrix}1\\\lambda\\\lambda^2\end{bmatrix}=0,\\
\left(\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+\lambda\begin{bmatrix}0&0&0\\-1&0&0\\0&0&0\end{bmatrix}+\mu\begin{bmatrix}0&-1&0\\0&0&0\\0&0&0\end{bmatrix}+\nu\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\right)\begin{bmatrix}1\\\lambda\\\lambda^2\end{bmatrix}=0,\\
\left(\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+\mu\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}+\sigma\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\right)\begin{bmatrix}1\\\mu\\\mu^2\end{bmatrix}=0. \quad (22)
\end{gathered}
$$

It can be seen that if $\big((\lambda,\mu),\ x_1\otimes x_2\big)$ is an eigenpair of the $\mathbb{CTEP}$ defined in (9), then

$$
\left((\lambda,\mu,\lambda^2,\lambda\mu,\mu^2,\lambda^3,\lambda^2\mu,\lambda\mu^2,\mu^3),x_1\otimes x_2\otimes\begin{bmatrix}1\\\lambda\\\lambda^2\end{bmatrix}\otimes\begin{bmatrix}1\\\lambda\\\lambda^2\end{bmatrix}\otimes\begin{bmatrix}1\\\mu\\\mu^2\end{bmatrix}\otimes\begin{bmatrix}1\\\mu\\\mu^2\end{bmatrix}\otimes\begin{bmatrix}1\\\lambda\\\lambda^2\end{bmatrix}\otimes\begin{bmatrix}1\\\lambda\\\lambda^2\end{bmatrix}\otimes\begin{bmatrix}1\\\mu\\\mu^2\end{bmatrix}\right)
$$

is an eigenpair of (22). For $\mathbb{L}9\mathbb{EP}$, the associated system of $\mathbb{GEP}$ becomes $\Delta_j u=\lambda_j\Delta_0 u;\ j:=1,\dots,9$ (Atkinson, 1972). Khazanov linearization and the Standard linearization produce a singular $\mathbb{L}2\mathbb{EP}$. On the other hand, the linearization method produces a nonsingular $\mathbb{L}9\mathbb{EP}$, which can be shown by the following lemma.

Lemma 8. The homogeneous version of the nine-parameter problem defined in (22) is nonsingular.

Proof. We consider,

$$
\lambda=\frac{\tilde{\lambda}}{\tilde{\kappa}},\mu=\frac{\tilde{\mu}}{\tilde{\kappa}},\gamma=\frac{\tilde{\gamma}}{\tilde{\kappa}},\delta=\frac{\tilde{\delta}}{\tilde{\kappa}},\eta=\frac{\tilde{\eta}}{\tilde{\kappa}},\nu=\frac{\tilde{\nu}}{\tilde{\kappa}},\sigma=\frac{\tilde{\sigma}}{\tilde{\kappa}}
$$

Multiplying each Equation of (22) by $\tilde{\kappa}$, the homogeneous version of the problem is obtained as,

$$
\begin{aligned}
&\det\big(\tilde{\alpha}A_{30}+\tilde{\beta}A_{21}+\tilde{\gamma}A_{12}+\tilde{\delta}A_{03}+\tilde{\eta}A_{20}+\tilde{\nu}A_{11}+\tilde{\sigma}A_{02}+\tilde{\lambda}A_{10}+\tilde{\mu}A_{01}+\tilde{\kappa}A_{00}\big)=0,\\
&\det\big(\tilde{\alpha}B_{30}+\tilde{\beta}B_{21}+\tilde{\gamma}B_{12}+\tilde{\delta}B_{03}+\tilde{\eta}B_{20}+\tilde{\nu}B_{11}+\tilde{\sigma}B_{02}+\tilde{\lambda}B_{10}+\tilde{\mu}B_{01}+\tilde{\kappa}B_{00}\big)=0,\\
&\tilde{\alpha}\tilde{\kappa}-\widetilde{\lambda^3}=0,\\
&\tilde{\beta}\tilde{\kappa}-\widetilde{\lambda^2}\tilde{\mu}=0,\\
&\tilde{\gamma}\tilde{\kappa}-\tilde{\lambda}\widetilde{\mu^2}=0,\\
&\tilde{\delta}\tilde{\kappa}-\widetilde{\mu^3}=0,\\
&\tilde{\eta}\tilde{\kappa}-\widetilde{\lambda^2}=0,\\
&\tilde{\nu}\tilde{\kappa}-\tilde{\lambda}\tilde{\mu}=0,\\
&\tilde{\sigma}\tilde{\kappa}-\widetilde{\mu^2}=0. \qquad (23)
\end{aligned}
$$

DOI:https//doi.org/10.22452/mjs.vol44no1.9
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Now consider $(\tilde{\kappa},\tilde{\lambda},\tilde{\mu},\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta},\tilde{\eta},\tilde{\nu},\tilde{\sigma})$ to be an eigenvalue of (23) such that $\tilde{\alpha}=0$. Then, the equations defined in (23) transform into

$$\begin{aligned}
&\det(\tilde{\beta}A_{21}+\tilde{\gamma}A_{12}+\tilde{\delta}A_{03}+\tilde{\eta}A_{20}+\tilde{\nu}A_{11}+\tilde{\sigma}A_{02}+\tilde{\lambda}A_{10}+\tilde{\mu}A_{01}+\tilde{\kappa}A_{00})=0,\\
&\det(\tilde{\beta}B_{21}+\tilde{\gamma}B_{12}+\tilde{\delta}B_{03}+\tilde{\eta}B_{20}+\tilde{\nu}B_{11}+\tilde{\sigma}B_{02}+\tilde{\lambda}B_{10}+\tilde{\mu}B_{01}+\tilde{\kappa}B_{00})=0,\\
&-\widetilde{\lambda^3}=0,\\
&\tilde{\beta}\tilde{\kappa}-\widetilde{\lambda^2}\tilde{\mu}=0,\\
&\tilde{\gamma}\tilde{\kappa}-\tilde{\lambda}\widetilde{\mu^2}=0,\\
&\tilde{\delta}\tilde{\kappa}-\widetilde{\mu^3}=0,\\
&\tilde{\eta}\tilde{\kappa}-\widetilde{\lambda^2}=0,\\
&\tilde{\nu}\tilde{\kappa}-\tilde{\lambda}\tilde{\mu}=0,\\
&\tilde{\sigma}\tilde{\kappa}-\widetilde{\mu^2}=0.
\end{aligned} \tag{24}$$

From the third Equation, we have $\tilde{\lambda}=0$. After substituting its value in the subsequent equations of (24) we obtain

$$\tilde{\beta}\tilde{\kappa}=0,\ \tilde{\gamma}\tilde{\kappa}=0,\ \tilde{\delta}\tilde{\kappa}=\mu^3,\ \tilde{\eta}\tilde{\kappa}=0,\ \tilde{\nu}\tilde{\kappa}=0,\ \tilde{\sigma}\tilde{\kappa}=\mu^2.$$

For all of these conditions, two cases may arise.

3. If $\tilde{\kappa}=0$, then the sixth and the last Equation in (24) give $\mu=0$. Thus, the remaining equations become,

$$\begin{aligned}
&\det(\tilde{\beta}A_{21}+\tilde{\gamma}A_{12}+\tilde{\delta}A_{03}+\tilde{\eta}A_{20}+\tilde{\nu}A_{11}+\tilde{\sigma}A_{02})=0,\\
&\det(\tilde{\beta}B_{21}+\tilde{\gamma}B_{12}+\tilde{\delta}B_{03}+\tilde{\eta}B_{20}+\tilde{\nu}B_{11}+\tilde{\sigma}B_{02})=0,
\end{aligned}$$

which has no solution in the general case.

4. If $\tilde{\kappa}\neq 0$, then for each of the above conditions, we obtain

$$\tilde{\beta}=0,\ \tilde{\gamma}=0,\ \tilde{\delta}=\frac{\widetilde{\mu^3}}{\tilde{\kappa}},\ \tilde{\eta}=0,\ \tilde{\nu}=0,\ \tilde{\sigma}=\frac{\widetilde{\mu^2}}{\tilde{\kappa}}.$$

Considering the value of δ, we obtain the above system (24) as follows,

$$\begin{aligned}
&\det\left(\frac{\widetilde{\mu^3}}{\tilde{\kappa}}A_{03}+\frac{\widetilde{\mu^2}}{\tilde{\kappa}}A_{02}+\frac{\tilde{\mu}}{\tilde{\kappa}}A_{01}+A_{00}\right)=0\\
&\det\left(\frac{\widetilde{\mu^3}}{\tilde{\kappa}}B_{03}+\frac{\widetilde{\mu^2}}{\tilde{\kappa}}B_{02}+\frac{\tilde{\mu}}{\tilde{\kappa}}B_{01}+B_{00}\right)=0
\end{aligned}$$

This has no solutions in general. Thus, the problem defined in Equation (22) does not have an eigenvalue with $\alpha=0$. It follows Theorem 3, where the operator matrix Δ_3 is nonsingular. Similarly, the operator matrices Δ_i for $i=4,\dots,9$ are nonsingular. □

5. Ranks of Delta Matrices

In Muhič & Plestenjak (2010), the Kronecker structures for the Delta matrices of the $\mathbb{QTEP}$ have been discussed extensively to prove the similarity between the eigenvalues of the linearized form and the original nonlinear form. Due to the complex Kronecker structures for the standard linearization (15) of $\mathbb{CTEP}$ defined in (9), they did not attempt to prove their ranks and related theory. The ranks of the Delta matrices will help us discover interesting structures and prove the singularity of the $\mathbb{L}2\mathbb{EP}$ defined in (15). These results can be viewed as a continuing series of proofs demonstrating that, in accordance with Theorem 17 in Muhič & Plestenjak (2010), the eigenvalues of (15) and (9) are identical.

Determining the rank of Delta matrices is crucial to understanding the nature and number of eigenvalues. Through rank determination, we show that all linear combinations of the corresponding operator determinants are singular. When determining the ranks of the Delta matrices, it is more straightforward to work with the Tracy-Singh product rather than the Kronecker product, as demonstrated by Definitions 2, 3, and 4.

DOI:https//doi.org/10.22452/mjs.vol44no1.9
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Finding the rank of Δ_0

Consider the operator determinant Δ_0 defined in (11). The structures of sub-matrices of Δ_0 become

$$\mathbb{L}_0^{(1)} = \begin{bmatrix} A_{00} & A_{10} & A_{01} & A_{20} & A_{11} & A_{02} \\ 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix}, \quad \mathbb{L}_1^{(1)} = \begin{bmatrix} 0 & 0 & 0 & A_{30} & A_{21} & A_{12} \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbb{L}_2^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & A_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}, \mathbb{L}_0^{(2)} = \begin{bmatrix} B_{00} & B_{10} & B_{01} & B_{20} & B_{11} & B_{02} \\ 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix},$$

$$\mathbb{L}_1^{(2)} = \begin{bmatrix} 0 & 0 & 0 & B_{30} & B_{21} & B_{12} \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbb{L}_2^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & B_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}.$$

If we apply the Tracy-Singh reordering to Δ_0, we obtain

$$TSR(\Delta_0) = \begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix},$$

where $S \in \mathbb{C}^{6n^2 \times 18n^2}$ and $T \in \mathbb{C}^{30n^2 \times 18n^2}$. The block structure representation of S is found to be of the form $S = [A \quad B \quad C]$; where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & A_{30} \otimes B_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ A_{30} \otimes I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{30} \otimes I & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & A_{21} \otimes B_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ A_{21} \otimes I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{21} \otimes I & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & -A_{03} \otimes B_{30} & -A_{03} \otimes B_{21} & A_{12} \otimes B_{03} - A_{03} \otimes B_{12} \\ -A_{03} \otimes I & 0 & 0 & 0 & 0 & 0 \\ A_{12} \otimes I & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_{03} \otimes I & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_{03} \otimes I & 0 & 0 & 0 \\ 0 & 0 & A_{12} \otimes I & 0 & 0 & 0 \end{bmatrix}$$

Similarly, the block representation of T can be determined as

$$T = \begin{bmatrix} D & 0 & 0 \\ E & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & G \\ 0 & 0 & H \end{bmatrix},$$

where $D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I \otimes B_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I \otimes I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I \otimes I & 0 & 0 & 0 \end{bmatrix},$

DOI:https//doi.org/10.22452/mjs.vol44no1.9
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$$E=\begin{bmatrix}0&0&0&-I\otimes B_{30}&-I\otimes B_{21}&-I\otimes B_{12}\\-I\otimes I&0&0&0&0&0\\0&0&0&0&0&0\\0&-I\otimes I&0&0&0&0\\0&0&-I\otimes I&0&0&0\\0&0&0&0&0&0\end{bmatrix},$$

$$F=\begin{bmatrix}0&0&0&0&0&I\otimes B_{03}\\0&0&0&0&0&0\\I\otimes I&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&I\otimes I&0&0&0\end{bmatrix},$$

$$G=\begin{bmatrix}0&0&0&0&0&I\otimes B_{03}\\0&0&0&0&0&0\\I\otimes I&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&I\otimes I&0&0&0\end{bmatrix},$$

$$H=\begin{bmatrix}0&0&0&-I\otimes B_{30}&-I\otimes B_{21}&-I\otimes B_{12}\\-I\otimes I&0&0&0&0&0\\0&0&0&0&0&0\\0&-I\otimes I&0&0&0&0\\0&0&-I\otimes I&0&0&0\\0&0&0&0&0&0\end{bmatrix}.$$

Considering the matrices $A_{03}, A_{30}, A_{12}, A_{21}, B_{30}$ and B_{03}as nonsingular, we conclude that the rank of S is $6n^2$ and the rank of T is$14n^2$. Thus, the rank of Δ_0 can be found as $20n^2 < 36n^2$.

Since Δ_0 is singular, the associated $\mathbb{L}2\mathbb{EP}$ defined in (15) is also singular. Using a similar technique, the singularity of Khazanov linearization can be proven as well.

Finding the rank of Δ_1

Consider a related problem

$\mathbb{P}_1'(\lambda,\mu) = A_{00} + \mu A_{01} + \mu^2 A_{02} + \mu^3 A_{03}$,

$\mathbb{P}_2'(\lambda,\mu) = \mathbb{P}_2(\lambda,\mu) = \lambda^3 B_{30} + \lambda^2\mu B_{21} + \lambda\mu^2 B_{12} + \mu^3 B_{03} + \lambda^2 B_{20} + \lambda\mu B_{11} + \mu^2 B_{02} + \lambda B_{10} + \mu B_{01} + B_{00}$ (25)

By linearizing $\mathbb{P}_1'(\lambda,\mu)$, we get

$$L_1'(\lambda,\mu)=\begin{bmatrix}A_{00}&A_{01}&A_{02}\\0&0&-I\\0&-I&0\end{bmatrix}+\mu\begin{bmatrix}0&0&A_{03}\\0&I&0\\I&0&0\end{bmatrix}.$$

$\mathbb{P}_2'(\lambda,\mu)$ is linearized as in $\mathbb{P}_2(\lambda,\mu)$.

$$L_2'(\lambda,\mu)=\begin{bmatrix}B_{00}&B_{10}&B_{01}&B_{20}&B_{11}&B_{02}\\0&-I&0&0&0&0\\0&0&-I&0&0&0\\0&0&0&-I&0&0\\0&0&0&0&-I&0\\0&0&0&0&0&-I\end{bmatrix}+\lambda\begin{bmatrix}0&0&0&B_{30}&B_{21}&B_{12}\\I&0&0&0&0&0\\0&0&0&0&0&0\\0&I&0&0&0&0\\0&0&I&0&0&0\\0&0&0&0&0&0\end{bmatrix}+\mu\begin{bmatrix}0&0&0&0&0&B_{03}\\0&0&0&0&0&0\\I&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&I&0&0&0\end{bmatrix}$$

Now, these two linearizations can be rewritten in the form of (10), where the coefficient matrices are

$$\mathbb{L}_0^{(1)}=\begin{bmatrix}A_{00}&A_{01}&A_{02}\\0&0&-I\\0&-I&0\end{bmatrix},\quad \mathbb{L}_1^{(1)}=0,\quad \mathbb{L}_2^{(1)}=\begin{bmatrix}0&0&A_{03}\\0&I&0\\I&0&0\end{bmatrix}$$

and

$$\mathbb{L}_0^{(2)}=\begin{bmatrix}B_{00}&B_{10}&B_{01}&B_{20}&B_{11}&B_{02}\\0&-I&0&0&0&0\\0&0&-I&0&0&0\\0&0&0&-I&0&0\\0&0&0&0&-I&0\\0&0&0&0&0&-I\end{bmatrix},\quad \mathbb{L}_1^{(2)}=\begin{bmatrix}0&0&0&B_{30}&B_{21}&B_{12}\\I&0&0&0&0&0\\0&0&0&0&0&0\\0&I&0&0&0&0\\0&0&I&0&0&0\\0&0&0&0&0&0\end{bmatrix}$$

DOI:https//doi.org/10.22452/mjs.vol44no1.9
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$$\mathbb{L}_2^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & B_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}.$$

Then,

$$\Delta_1' = \mathbb{L}_2^{(1)} \otimes \mathbb{L}_0^{(2)} - \mathbb{L}_0^{(1)} \otimes \mathbb{L}_2^{(2)}.$$

This shows that Δ_1' is non-singular.

Let Δ_1' be singular. By Theorem 6, the system (10) has an eigenvalue $(\eta_0, 0, \eta_2)$ such that $(\eta_0, \eta_2) \neq (0,0)$. As in the general case, $\mathbb{L}_2^{(2)}$ is nonsingular, so $\eta_0 \neq 0$, which indicates that the original problem has an eigenvalue of the form $(0, \mu)$. Therefore, Δ_1' has to be nonsingular.

By the Tracy-Singh product of Δ_1, we have

$$TSP(\Delta_1) = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ 0 & S_{22} & 0 & 0 & 0 & 0 \\ S_{31} & 0 & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & S_{63} & 0 & 0 & S_{66} \end{bmatrix};$$

where

$$S_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -A_{00} \otimes B_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -A_{00} \otimes I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_{00} \otimes I & 0 & 0 & 0 \end{bmatrix},$$

$$S_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -A_{10} \otimes B_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -A_{10} \otimes I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_{10} \otimes I & 0 & 0 & 0 \end{bmatrix},$$

$$S_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -A_{01} \otimes B_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -A_{01} \otimes I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_{01} \otimes I & 0 & 0 & 0 \end{bmatrix},$$

$$S_{14} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -A_{20} \otimes B_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -A_{20} \otimes I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_{20} \otimes I & 0 & 0 & 0 \end{bmatrix},$$

$$S_{15} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -A_{11} \otimes B_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -A_{11} \otimes I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_{11} \otimes I & 0 & 0 & 0 \end{bmatrix},$$

DOI:https//doi.org/10.22452/mjs.vol44no1.9
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$$S_{16}=\begin{bmatrix} A_{03}\otimes B_{00} & A_{03}\otimes B_{10} & A_{03}\otimes B_{01} & A_{03}\otimes B_{20} & A_{03}\otimes B_{11} & A_{03}\otimes B_{02}-A_{02}\otimes B_{03} \\ 0 & -A_{03}\otimes I & 0 & 0 & 0 & 0 \\ -A_{02}\otimes I & 0 & -A_{03}\otimes I & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_{03}\otimes I & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_{03}\otimes I & 0 \\ 0 & 0 & -A_{02}\otimes I & 0 & 0 & -A_{03}\otimes I \end{bmatrix},$$

$$S_{31}=S_{63}=\begin{bmatrix} I\otimes B_{00} & I\otimes B_{10} & I\otimes B_{01} & I\otimes B_{20} & I\otimes B_{11} & I\otimes B_{02} \\ 0 & -I\otimes I & 0 & 0 & 0 & 0 \\ 0 & 0 & -I\otimes I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I\otimes I & 0 & 0 \\ 0 & 0 & 0 & 0 & -I\otimes I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I\otimes I \end{bmatrix},$$

$$S_{22}=S_{33}=S_{44}=S_{55}=S_{66}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I\otimes B_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I\otimes I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I\otimes I & 0 & 0 & 0 \end{bmatrix}.$$

Now, by the Tracy-Singh reordering of Δ_1', we obtain

$$TSR(\Delta_1')=\begin{pmatrix} S_{11} & S_{13} & S_{16} \\ S_{31} & S_{33} & S_{36} \\ S_{61} & S_{63} & S_{66} \end{pmatrix}.$$

If we perform Tracy-Singh reordering in Δ_1, then we have

$$TSR(\Delta_1)=\begin{bmatrix} S_{11} & S_{13} & S_{16} & S_{14} & S_{15} & S_{12} \\ S_{31} & S_{33} & 0 & 0 & 0 & 0 \\ 0 & S_{63} & S_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{22} \end{bmatrix}.$$

Since $TSR(\Delta_1')$ is nonsingular, the remaining diagonal block entries S_{22}, S_{44}, and $S_{55}$of $TSR(\Delta_1)$ yield a maximal rank $9n^2$, assuming B_{03} is nonsingular. Thus, this shows that the matrix Δ_1 is of rank $27n^2$.

Finding the rank of Δ_2

Consider a related problem, where

$\mathbb{P}_1'(\lambda,\mu)=A_{00}+\lambda A_{10}+\lambda^2 A_{20}+\lambda^3 A_{30}$,

$\mathbb{P}_2'(\lambda,\mu)=\mathbb{P}_1(\lambda,\mu)=\lambda^3 B_{30}+\lambda^2\mu B_{21}+\lambda\mu^2 B_{12}+\mu^3 B_{03}+\lambda^2 B_{20}+\lambda\mu B_{11}+\mu^2 B_{02}+\lambda B_{10}+\mu B_{01}+B_{00}$ (26)

By linearizing $\mathbb{P}_1'(\lambda,\mu)$, we obtain

$$L_1'(\lambda,\mu)=\begin{bmatrix} A_{00} & A_{10} & A_{20} \\ 0 & 0 & -I \\ 0 & -I & 0 \end{bmatrix}+\lambda\begin{bmatrix} 0 & 0 & A_{30} \\ 0 & I & 0 \\ I & 0 & 0 \end{bmatrix}.$$

$\mathbb{P}_2'(\lambda,\mu)$ is linearized as in $\mathbb{P}_2(\lambda,\mu)$.

$$L_2'(\lambda,\mu)=\begin{bmatrix} B_{00} & B_{10} & B_{01} & B_{20} & B_{11} & B_{02} \\ 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix}+\lambda\begin{bmatrix} 0 & 0 & 0 & B_{30} & B_{21} & B_{12} \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}+\mu\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & B_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}.$$

Now, these two linearizations can be rewritten in the form of (10), where the coefficient matrices become

$$\mathbb{L}_0^{(1)}=\begin{bmatrix} A_{00} & A_{10} & A_{20} \\ 0 & 0 & -I \\ 0 & -I & 0 \end{bmatrix},\quad \mathbb{L}_1^{(1)}=\begin{bmatrix} 0 & 0 & A_{30} \\ 0 & I & 0 \\ I & 0 & 0 \end{bmatrix},\quad \mathbb{L}_2^{(1)}=0,$$

and

DOI:https//doi.org/10.22452/mjs.vol44no1.9
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$$\mathbb{L}_0^{(2)} = \begin{bmatrix} B_{00} & B_{10} & B_{01} & B_{20} & B_{11} & B_{02} \\ 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix}, \quad \mathbb{L}_1^{(2)} = \begin{bmatrix} 0 & 0 & 0 & B_{30} & B_{21} & B_{12} \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbb{L}_2^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & B_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}.$$

Then,

$$\Delta_2' = \mathbb{L}_0^{(1)} \otimes \mathbb{L}_1^{(2)} - \mathbb{L}_1^{(1)} \otimes \mathbb{L}_0^{(2)}.$$

We claim that Δ_2' of the related problem is nonsingular.

Let us consider Δ_2' to be singular. By Theorem 6, the related system (10) has an eigenvalue $(\eta_0, \eta_1, 0)$ such that $(\eta_0, \eta_1) \neq (0,0)$. As in the general case, the matrix $\mathbb{L}_1^{(2)}$ is nonsingular, so $\eta_0 \neq 0$, which indicates that the original problem has an eigenvalue of the form $(\lambda, 0)$. So Δ_2' must be nonsingular.
By the Tracy-Singh product of Δ_2, we have,

$$TSP(\Delta_2) = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & S_{33} & 0 & 0 & 0 \\ 0 & S_{42} & 0 & S_{44} & 0 & 0 \\ 0 & 0 & S_{53} & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix},$$

where

$$S_{11} = \begin{bmatrix} 0 & 0 & 0 & A_{00} \otimes B_{30} & A_{00} \otimes B_{21} & A_{00} \otimes B_{12} \\ A_{00} \otimes I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{00} \otimes I & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{00} \otimes I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S_{12} = \begin{bmatrix} 0 & 0 & 0 & A_{10} \otimes B_{30} & A_{10} \otimes B_{21} & A_{10} \otimes B_{12} \\ A_{10} \otimes I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{10} \otimes I & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{10} \otimes I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S_{13} = \begin{bmatrix} 0 & 0 & 0 & A_{01} \otimes B_{30} & A_{01} \otimes B_{21} & A_{01} \otimes B_{12} \\ A_{01} \otimes I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{01} \otimes I & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{01} \otimes I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S_{14} = \begin{bmatrix} -A_{30} \otimes B_{00} & -A_{30} \otimes B_{10} & -A_{30} \otimes B_{01} & A_{20} \otimes B_{30} - A_{30} \otimes B_{20} & A_{20} \otimes B_{21} - A_{30} \otimes B_{11} & A_{20} \otimes B_{12} - A_{30} \otimes B_{02} \\ A_{20} \otimes I & A_{30} \otimes I & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{30} \otimes I & 0 & 0 & 0 \\ 0 & A_{20} \otimes I & 0 & A_{30} \otimes I & 0 & 0 \\ 0 & 0 & A_{20} \otimes I & 0 & A_{30} \otimes I & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{30} \otimes I \end{bmatrix},$$

$$S_{15} = \begin{bmatrix} -A_{21} \otimes B_{00} & -A_{21} \otimes B_{10} & -A_{21} \otimes B_{01} & A_{11} \otimes B_{21} - A_{30} \otimes B_{20} & A_{11} \otimes B_{21} - A_{21} \otimes B_{11} & A_{11} \otimes B_{12} - A_{21} \otimes B_{02} \\ A_{11} \otimes I & A_{21} \otimes I & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{21} \otimes I & 0 & 0 & 0 \\ 0 & A_{11} \otimes I & 0 & A_{21} \otimes I & 0 & 0 \\ 0 & 0 & A_{11} \otimes I & 0 & A_{21} \otimes I & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{21} \otimes I \end{bmatrix},$$

DOI:https//doi.org/10.22452/mjs.vol44no1.9
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$$S_{16} = \begin{bmatrix} -A_{12}\otimes B_{00} & -A_{12}\otimes B_{10} & -A_{12}\otimes B_{01} & A_{02}\otimes B_{21} - A_{30}\otimes B_{20} & A_{02}\otimes B_{21} - A_{12}\otimes B_{11} & A_{02}\otimes B_{12} - A_{12}\otimes B_{02} \\ A_{02}\otimes I & A_{12}\otimes I & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{12}\otimes I & 0 & 0 & 0 \\ 0 & A_{02}\otimes I & 0 & A_{12}\otimes I & 0 & 0 \\ 0 & 0 & A_{02}\otimes I & 0 & A_{12}\otimes I & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{12}\otimes I \end{bmatrix},$$

$$S_{21} = \begin{bmatrix} -I\otimes B_{00} & -I\otimes B_{10} & -I\otimes B_{01} & -I\otimes B_{20} & -I\otimes B_{11} & -I\otimes B_{02} \\ 0 & I\otimes I & 0 & 0 & 0 & 0 \\ 0 & 0 & I\otimes I & 0 & 0 & 0 \\ 0 & 0 & 0 & I\otimes I & 0 & 0 \\ 0 & 0 & 0 & 0 & I\otimes I & 0 \\ 0 & 0 & 0 & 0 & 0 & I\otimes I \end{bmatrix},$$

$$S_{22} = S_{33} = S_{44} = S_{55} = S_{66} = \begin{bmatrix} 0 & 0 & 0 & -I\otimes B_{30} & -I\otimes B_{21} & -I\otimes B_{12} \\ -I\otimes I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I\otimes I & 0 & 0 & 0 & 0 \\ 0 & 0 & -I\otimes I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S_{42} = \begin{bmatrix} -I\otimes B_{00} & -I\otimes B_{10} & -I\otimes B_{01} & -I\otimes B_{20} & -I\otimes B_{11} & -I\otimes B_{02} \\ 0 & I\otimes I & 0 & 0 & 0 & 0 \\ 0 & 0 & I\otimes I & 0 & 0 & 0 \\ 0 & 0 & 0 & I\otimes I & 0 & 0 \\ 0 & 0 & 0 & 0 & I\otimes I & 0 \\ 0 & 0 & 0 & 0 & 0 & I\otimes I \end{bmatrix},$$

$$S_{53} = \begin{bmatrix} -I\otimes B_{00} & -I\otimes B_{10} & -I\otimes B_{01} & -I\otimes B_{20} & -I\otimes B_{11} & -I\otimes B_{02} \\ 0 & I\otimes I & 0 & 0 & 0 & 0 \\ 0 & 0 & I\otimes I & 0 & 0 & 0 \\ 0 & 0 & 0 & I\otimes I & 0 & 0 \\ 0 & 0 & 0 & 0 & I\otimes I & 0 \\ 0 & 0 & 0 & 0 & 0 & I\otimes I \end{bmatrix}.$$

By using the Tracy-Singh reordering of Δ_2', we have

$$\mathrm{TSR}(\Delta_2') = \begin{pmatrix} S_{11} & S_{12} & S_{14} \\ S_{21} & S_{22} & S_{24} \\ S_{41} & S_{42} & S_{44} \end{pmatrix}.$$

Again, with the Tracy-Singh reordering of Δ_2, we obtain

$$TSR(\Delta_2) = \begin{bmatrix} S_{11} & S_{12} & S_{14} & S_{13} & S_{15} & S_{16} \\ S_{12} & S_{22} & 0 & 0 & 0 & 0 \\ 0 & S_{42} & S_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & S_{53} & 0 & S_{66} \end{bmatrix}.$$

Since $TSR(\Delta_2')$ is nonsingular, the remaining block entries S_{33}, S_{53}, S_{55} and S_{66} of $TSR(\Delta_2)$ give us a maximal rank of $12n^2$, assuming B_{ij} are nonsingular. Thus, this shows that the matrix Δ_2 is of rank $30n^2 < 36n^2$.

In the next section, we consider a randomly generated $\mathbb{CTEP}$, where the coefficients matrices are taken as real diagonal matrices. Then, we compare the eigenvalues obtained through the standard linearization and the Khazanov linearization processes. The results obtained in the case of the standard linearization via MatParEig Package (Muhič & Plestenjak, 2010) are considered the correct ones. On this basis, the approximation for the eigenvalues are found via an algorithm designed in MATLAB.

The linearization-like method reduced $\mathbb{CTEP}$ into a nine-parameter linear problem, the $\mathbb{L}9\mathbb{EP}$. It requires more computational time to find the numerical solution due to an increase in the number of linear equations. Moreover, the dimensions of the corresponding Δ_i matrices induced via Kronecker product also increase, making them sparse and computationally inefficient. Therefore, the linearization-like method is slower than the other two methods. The analytical comparison of the linearization-like method with the other two methods is omitted here due to the time complexity issues in calculating the corresponding Kronecker structure.

DOI:https//doi.org/10.22452/mjs.vol44no1.9
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6. Numerical Example

Consider a $\mathbb{CTEP}$,

$$\mathbb{P}_1(\lambda,\mu)x_1 = (\lambda^3 A_{30} + \lambda^2\mu A_{21} + \lambda\mu^2 A_{12} + \mu^3 A_{03} + \lambda^2 A_{20} + \lambda\mu A_{11} + \mu^2 A_{02} + \lambda A_{10} + \mu A_{01} + A_{00})x_1 = 0;$$

$$\mathbb{P}_2(\lambda,\mu)x_2 = (\lambda^3 B_{30} + \lambda^2\mu B_{21} + \lambda\mu^2 B_{12} + \mu^3 B_{03} + \lambda^2 B_{20} + \lambda\mu B_{11} + \mu^2 B_{02} + \lambda B_{10} + \mu B_{01} + B_{00})x_2 = 0;$$

with randomly generated diagonal matrices, A_{ij} and B_{ij} of order 2×2.

$$A_{00} = \begin{bmatrix} 0.8147 & 0 \\ 0 & 0.9134 \end{bmatrix}, A_{10} = \begin{bmatrix} 0.6324 & 0 \\ 0 & 0.5469 \end{bmatrix}, A_{01} = \begin{bmatrix} 0.9575 & 0 \\ 0 & 0.9706 \end{bmatrix},$$

$$A_{20} = \begin{bmatrix} 0.9572 & 0 \\ 0 & 0.1419 \end{bmatrix}, A_{11} = \begin{bmatrix} 0.4218 & 0 \\ 0 & 0.9595 \end{bmatrix}, A_{02} = \begin{bmatrix} 0.6557 & 0 \\ 0 & 0.9340 \end{bmatrix},$$

$$A_{30} = \begin{bmatrix} 0.6787 & 0 \\ 0 & 0.3922 \end{bmatrix}, A_{21} = \begin{bmatrix} 0.6555 & 0 \\ 0 & 0.0318 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.2769 & 0 \\ 0 & 0.8235 \end{bmatrix},$$

$$A_{03} = \begin{bmatrix} 0.6948 & 0 \\ 9 & 0.0344 \end{bmatrix},$$

$$B_{00} = \begin{bmatrix} 0.1869 & 0 \\ 0 & 0.6463 \end{bmatrix}, B_{10} = \begin{bmatrix} 0.7094 & 0 \\ 0 & 0.6797 \end{bmatrix}, B_{01} = \begin{bmatrix} 0.6551 & 0 \\ 0 & 0.4984 \end{bmatrix},$$

$$B_{20} = \begin{bmatrix} 0.9597 & 0 \\ 0 & 0.2238 \end{bmatrix}, B_{11} = \begin{bmatrix} 0.7513 & 0 \\ 0 & 0.6991 \end{bmatrix}, B_{02} = \begin{bmatrix} 0.8909 & 0 \\ 0 & 0.1386 \end{bmatrix},$$

$$B_{30} = \begin{bmatrix} 0.1493 & 0 \\ 0 & 0.2543 \end{bmatrix}, B_{21} = \begin{bmatrix} 0.8143 & 0 \\ 0 & 0.3500 \end{bmatrix}, B_{12} = \begin{bmatrix} 0.1966 & 0 \\ 0 & 0.4377 \end{bmatrix},$$

$$B_{03} = \begin{bmatrix} 0.3517 & 0 \\ 0 & 0.5497 \end{bmatrix}.$$

To compare the numerical results obtained from Standard linearization and Khanzadeh linearization, we used the MultiParEig toolbox developed by Plestenjak (2023) on a Windows 11 operating system with an AMD Ryzen 5 5500U 2.10 GHz processor. The results are shown below.

Table 1

Standard Linearization		Khazanov Linearization	
λ	μ	λ	μ
1.0092+0.0000i	-1.4499+0.0000i	1.0092 + 0.0000i	-1.4499+0.0000i
-0.3421+0.0000i	-0.7805+0.0000i	-0.3421 +0.0000i	-0.7805+0.0000i
-0.3432±0.8405i	-0.9510±0.0087i	-0.3432±0.8405i	-0.9510±0.0087i
-1.3114±0.2309i	-0.9173±1.4685i	-1.3114±0.2309i	-0.9173±1.4685i
-1.0672±0.1223i	-0.3852±0.9591i	-1.0672±0.1223i	-0.3852±0.9591i
-0.7862+0.0000i	-0.5743 +0.0000i	-0.7862 +0.0000i	-0.5743+0.0000i
0.3349±0.0446i	-0.5398±0.8207i	0.3349±0.0446i	-0.5398±0.8207i
-1.0856+0.0000i	0.7836 + 0.0000i	-1.0854+0.0000i	0.7776 + 0.0000i
-1.5614±0.2974i	-0.0133±1.7493i	-1.5614±0.2974i	-0.0133±1.7493i
-1.0610±0.0111i	-0.0168±0.5738i	-1.0602±0.0109i	-0.0098±0.5834

DOI:https//doi.org/10.22452/mjs.vol44no1.9
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0.3768±0.6614i	0.3238±1.0721i	0.3768±0.6614i	0.3238±1.0721i
-0.1068±1.2095i	-0.0491±0.5295i	-0.4227±3.7101i	1.8669±0.5291i
-1.0224±0.5267i	-0.6265±0.5461i	0.2789±1.4519i	-1.1613±0.1619i
-1.0978+0.0000i	-0.7238 +0.0000i	-1.0978+0.0000i	-0.7238+0.0000i
1.1408 + 0.0000i	-90.1367+0.0000i	0.3469 + 0.0000i	-1.0288+0.0000i
-0.3051±0.7180i	0.1179±0.8142i	-0.4267±0.6475i	0.1663±0.9384i
-7.9598±6.4034i	-4.9047±5.8520i	0.6625±0.4573i	0.3300±1.3700i
0.8733±0.0768i	0.0933±1.6336i	0.5203±0.7182i	-1.0214±0.4763i
0.3425±1.2010i	-0.7668±0.1594i	0.2367±1.0175i	-0.7182±0.4862i
-0.0348±0.3753i	-0.8454±0.1057i	-1.0498±0.4383i	-0.4927±0.4840i
0.2789±1.4519i	-1.1613±0.1619i	-0.4080±1.4486i	0.2030±0.6769i

By Bézout's theorem, a $\mathbb{CTEP}$ has $9n^2$ eigenvalues; therefore, the problem considered above has $9n^2 = 9.2^2 = 36$ eigenvalues. Both methods calculate all the eigenvalues of the problem. The Standard linearization provides the exact eigenvalues. Through comparisons of the eigenvalues obtained by Khazanov linearization, we find that most eigenvalues, except for a few, match those obtained from the Standard linearization. In the MATLAB environment, the execution time for the Standard Linearization process is 1.267792 seconds, while Khazanov Linearization takes 0.431789 seconds. The Khazanov linearization is faster due to the small size of the coefficient matrix and the absence of antidiagonal elements. For small-ordered matrices, Khazanov linearization may be preferable to Standard Linearization. However, Standard linearization consistently yields better results when dealing with higher-order matrices.

7. Conclusion

We described the Kronecker canonical structures of $\mathbb{CTEP}$ obtained through different linearization processes, including standard linearization, Khazanov linearization, and transformation to $\mathbb{L9EP}$. These approaches can be used to find numerical solutions of $\mathbb{CTEP}$ by applying existing numerical methods to solve $\mathbb{LMEP}$s. We compared the first two singular linearizations through a numerical example. The calculation via $\mathbb{L9EP}$ is omitted here due to its higher computational time requirements. The Kronecker structures of Δ_i matrices for i:=0:2 have not been extensively studied because of their complex structures in $\mathbb{CTEP}$. These structures and their ranks can assist in developing proofs for the number of manifolds via algebraic geometry in various methods (Dong, (2022)). All results are novel and serve as a foundation for further study of the Delta matrices of $\mathbb{PTEP}$ of degree k.

Acknowledgment: The authors express their deep gratitude for the comments and valuable suggestions provided by the two anonymous reviewers, which considerably improved the paper. The authors also thank Dibrugarh University for providing all necessary facilities during their research work.

Funding: There are no funding sources.

Conflict of interest: All authors declare that there is no conflict of interest.

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DOI:https//doi.org/10.22452/mjs.vol44no1.9
Malaysian Journal of Science 44(1): 70-86 (March 2025)